Journal of Algebra, Number Theory: Advances and Applications
Volume 11, Number 1, 2014, Pages 33-47

CLOSED WEAK GENERALIZED SUPPLEMENTED MODULES

Y. TALEBI[1], T. AMOUZEGAR[2] and A. MAHMOUDI[1]

[1]Department of Mathematics
Faculty of Mathematical Sciences
University of Mazandaran
Babolsar
Iran
e-mail: talebi@umz.ac.ir

[2]Department of Mathematics
Quchan Institute of Engineering and Technology
Quchan
Iran

Abstract

A module M is called closed weak generalized supplemented if for every closed submodule N of M, there exists a submodule K of M such that $M = K + N$ and $K \cap N \leq Rad(M)$. We prove that for a distributive module $M = M_1 \oplus M_2$, M is closed weak generalized supplemented, if and only if M_1 and M_2 are closed weak generalized supplemented. We characterize nonsingular V-rings in which all nonsingular modules are closed weak generalized supplemented.

2010 Mathematics Subject Classification: 16D10, 16D80.

Keywords and phrases: closed submodule, closed weak generalized supplemented module, singular module.

Received January 20, 2014

1. Introduction

In this paper, all rings are associative rings with identity and all modules are unitary R-modules. Let R be a ring and M be an R-module. A submodule N of M is called *essential* in M (or M is an essential extension of N), if $N \cap A \neq 0$ for any nonzero submodule A of M. In this case, we denote $N \leq_e M$. Dually, a submodule N of M is called *small* in M, abbreviated $N \ll M$, if $M \neq N + T$ for any proper submodule T of M. A nonzero module M is called *hollow* if every proper submodule of M is small in M. M is called *local* if the sum of all proper submodules of M is also a proper submodule of M. Every local module is hollow. A closed submodule N of M, denoted by $N \leq_c M$, is a submodule, which has no proper essential extension in M. It is well known that if $L \leq_c N$ and $N \leq_c M$, then $L \leq_c M$ (see [3]). A module M is called an *extending* module (or CS-module) if every submodule of M is essential in a direct summand of M, or equivalently, every closed submodule of M is a direct summand of M.

Let K and N be submodules of M. K is called a *supplement* of N in M, if $M = K + N$ and K is minimal with respect to this property, or equivalently, $M = K + N$ and $K \cap N \ll K$. M is called a *supplemented module*, if every submodule of M has a supplement in M. Let K and N be submodules of M. K is called a *generalized supplement* of N in M when $M = N + K$ and $N \cap K \leq Rad(K)$. M is called a *generalized supplemented module* or briefly a GS-module in case each submodule N of M has a generalized supplement K in M (see [15]). Clearly, each supplement submodule is a generalized supplement submodule. Let K and N be submodules of M. K is called a *weak generalized supplement* of N in M, if $M = K + N$ and $K \cap N \leq Rad(M)$. A module M is called a *weak generalized supplemented module* or briefly a WGS-module, if each submodule of M has a weak generalized supplement in M (see [12]). Any supplemented module is a weak generalized supplemented module.

In this paper, we replace the condition of extending modules, i.e., any closed submodule is a direct summand, by the condition that any closed submodule has a weak generalized supplement. Thus, we generalize both extending modules and weak generalized supplemented modules to closed weak generalized supplemented modules.

In Section 2, we give a characterization of weak generalized supplemented modules. It is shown that finitely generated modules are weak generalized supplemented, if and only if they have finite hollow dimension.

In Section 3, we give the definition of closed weak generalized supplemented modules and show that any direct summand of a closed weak generalized supplemented module is closed weak generalized supplemented. We prove that if $M = M_1 \oplus M_2$ is a distributive module, then M is closed weak generalized supplemented if and only if each M_i $(i = 1, 2)$ is closed weak generalized supplemented.

In Section 4, we show that any nonsingular homomorphic image of a closed weak generalized supplemented module is a closed weak generalized supplemented module. We characterize nonsingular V-rings in which all nonsingular modules are closed weak generalized supplemented.

2. WGS-Modules

Proposition 2.1 ([7, Proposition 2.1])**.** *For a proper submodule N of M, the following statements are equivalent*:

(1) M / N *is semisimple*.

(2) *For every* $L \subseteq M$, *there exists a submodule* $K \subseteq M$ *such that* $L + K = M$ *and* $L \cap K \subseteq N$.

(3) *There exists a decomposition* $M = M_1 \oplus M_2$ *such that* M_1 *is semisimple*, $N \leq_e M_2$ *and* M_2 / N *is semisimple*.

Recall that a module M is said to be *semilocal*, if $M / Rad(M)$ is semisimple.

Corollary 2.2. *Let M be a module. Then the following statements are equivalent*:

(1) *M is semilocal*.

(2) *M is a WGS-module*.

(3) *There is a decomposition* $M = M_1 \oplus M_2$ *such that* M_1 *is semisimple*, $Rad(M) \leq_e M_2$ *and* $M_2 / Rad(M)$ *is semisimple*.

Proof. By Proposition 2.1. □

Corollary 2.3. *An R-module M with* $Rad(M) = 0$ *is a WGS-module*, *if and only if M is semisimple*.

Recall that a module M is said to have *finite hollow dimension*, if there exists an epimorphism $g : M \rightarrow \bigoplus_{i=1}^{n} H_i$, where all the H_i are hollow and $\operatorname{Ker} g \ll M$. Then n is called the hollow dimension of M and we write $h \dim(M) = n$.

The connection between the concepts of hollow dimension and the weak generalized supplemented is expressed in the following theorem:

Theorem 2.4. *Consider the following conditions for a module M*:

(i) *M has finite hollow dimension*.

(ii) *M is a WGS-module*.

(iii) *M is semilocal*.

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii). *If M is finitely generated*, *then* (iii) $\Rightarrow$ (i) *holds*.

Proof. (i) $\Rightarrow$ (ii) There is a small epimorphism $f : M \to \bigoplus_{i=1}^{n} H_i$ with hollow modules H_i. Since hollow modules are WGS-modules, by [12, Proposition 3.7], $\bigoplus_{i=1}^{n} H_i$ is a WGS-module. Since f is a small epimorphism, by [12, Proposition 3.2], M is a WGS-module.

(ii) $\Leftrightarrow$ (iii) By Corollary 2.2.

If M is finitely generated and (iii) holds, then M is a small cover of $M / Rad(M)$. Hence M has finite hollow dimension. □

Recall that the ring R is *semilocal*, if ${}_R R (\text{or } R_R)$ is a semilocal R-module.

Corollary 2.5. *For a ring R, the following statements are equivalent*:

(1) ${}_R R$ *is a WGS-module.*

(2) ${}_R R$ *has finite hollow dimension.*

(3) R *is semilocal.*

(4) R_R *has finite hollow dimension.*

(5) R_R *is a WGS-module.*

Proof. By Theorem 2.4 and using this fact that 'semilocal' is a left-right symmetric property. □

Theorem 2.6. *For any ring R, the following statements are equivalent*:

(1) R *is semilocal.*

(2) *Every right R-module is semilocal.*

(3) *Every right R-module is the direct sum of a semisimple module and a semilocal module with essential radical.*

(4) *Every right R-module is a WGS-module.*

(5) *Every finitely generated right R-module has finite hollow dimension.*

Proof. (1) $\Rightarrow$ (2) For any module M, there exists a set Λ and an epimorphism $f : R^{(\Lambda)} \to M$ with $f(Rad(R^{(\Lambda)})) \leq Rad(M)$ and $R^{(\Lambda)}/Rad(R^{(\Lambda)}) \simeq (R/J(R))^{(\Lambda)}$. Thus, we get an epimorphism $g : R^{(\Lambda)}/Rad(R^{(\Lambda)}) \to M/Rad(M)$. Hence M is semilocal.

(2) $\Rightarrow$ (1) It is clear.

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) By Corollary 2.2.

(4) $\Leftrightarrow$ (5) By Theorem 2.4. □

3. CWGS-Modules

Definition 3.1. A module M is called a *closed weak generalized supplemented module*, or briefly a CWGS-module, if for each closed submodule N of M, there exists a submodule K of M such that $M = K + N$ and $K \cap N \leq Rad(M)$.

Clearly, any WGS-module is a CWGS-module and any extending module is a CWGS-module. We have the following implications:

Hollow modules $\Rightarrow$ Generalized supplemented modules $\Rightarrow$ Weak generalized supplemented modules $\Rightarrow$ Closed weak generalized supplemented modules.

A CWGS-module need not be a WGS-module as the next example demonstrates.

Example 3.2. Let $\mathbb{Z}$ be the ring of all integers. Since $\mathbb{Z}$ is extending, it is a CWGS-module. But $\mathbb{Z}$ is not a WGS-module, since, for any $n \geq 2$, $n\mathbb{Z}$ has no weak generalized supplement in $\mathbb{Z}$.

Any direct summand of an extending module is extending [8]. For CWGS-modules, we have:

Proposition 3.3. *Let M be a CWGS-module. Then any direct summand of M is a CWGS-module.*

Proof. Let N be a direct summand of M and L be a closed submodule of N. Since N is closed in M, we have that L is closed in M. Then, there exists a submodule K of M such that $M = L + K$ and $L \cap K \leq Rad(M)$. Thus, $N = N \cap M = N \cap (L + K) = L + (N \cap K)$. Since N is a direct summand of M, then $L \cap (N \cap K) = N \cap (L \cap K) \leq N \cap Rad(M) = Rad(N)$ by [10, 41.1]. Thus N is a CWGS-module. □

For any ring R, any finite sum of WGS-modules is a WGS-module (see [12, Proposition 3.7]). For CWGS-modules, direct sums of CWGS-modules need not be CWGS-modules.

Example 3.4. Let $R = \mathbb{Z}[x]$, where $\mathbb{Z}$ is the ring of all integers. Set $M = R \oplus R$, then M is not extending (see [2]). As $Rad(M) = 0$, we see that M is not a CWGS-module.

Lemma 3.5. *Let N and L be submodules of a module M such that* $N + L$ *has a weak generalized supplement H in M and* $N \cap (H + L)$ *has a weak generalized supplement G in N. Then* $H + G$ *is a weak generalized supplement of L in M.*

Proof. Let H be a weak generalized supplement of $N + L$ in M and G be a weak generalized supplement of $N \cap (H + L)$ in N. So $M = (N + L) + H$, $(N + L) \cap H \leq Rad(M)$, $N = (N \cap (H + L)) + G$ and $(H + L) \cap G \leq Rad(N)$. We have $(H + G) \cap L \leq ((H + L) \cap G) + ((G + L) \cap H) \leq Rad(N) + Rad(M) \leq Rad(M)$. Thus, $H + G$ is a weak generalized supplement of L in M. □

Proposition 3.6. *Let $M = M_1 \oplus M_2$ such that each M_i $(i = 1, 2)$ is a CWGS-module. Suppose that $M_i \cap (M_j + L) \leq_c M_i$ and $M_j \cap (L + K) \leq_c M_j$, where K is a weak generalized supplement of $M_i \cap (M_j + L)$ in M_i, $i \neq j$, for any closed submodule L of M. Then M is a CWGS-module.*

Proof. Let $L \leq_c M$, then $M = M_1 + (M_2 + L)$ has a weak generalized supplement 0 in M. Since $M_1 \cap (M_2 + L) \leq_c M_1$ and M_1 is a CWGS-module, then there exists a submodule K of M_1 such that $M_1 = K + (M_1 \cap (M_2 + L))$ and $K \cap (M_1 \cap (M_2 + L)) = K \cap (M_2 + L) \leq Rad(M_1)$. By Lemma 3.5, K is a weak generalized supplement of $M_2 + L$ in M, i.e., $M = K + (M_2 + L)$. Since $M_2 \cap (K + L) \leq_c M_2$ and M_1 is a CWGS-module, then $M_2 \cap (K + L)$ has a weak generalized supplement J in M_2. Again by Lemma 3.5, $K + J$ is a weak generalized supplement of L in M. Hence M is a CWGS-module. □

Proposition 3.7. *Let $M = M_1 + M_2$, where M_1 is a CWGS-module and M_2 is any R-module. Suppose that for any closed submodule N of M, $N \cap M_1 \leq_c M_1$. Then M is a CWGS-module, if and only if every closed submodule N of M with M_2 not contained in N has a weak generalized supplement.*

Proof. $(\Rightarrow)$ It is clear.

$(\Leftarrow)$ Let $N \leq_c M$ such that $M_2 \subseteq N$. Then $M = M_1 + M_2 = M_1 + N$ and $M_1 + N$ has a weak generalized supplement 0. Since $N \cap M_1$ is closed in M_1 and M_1 is a CWGS-module, $N \cap M_1$ has a weak generalized supplement H in M_1. By Lemma 3.5, H is a weak generalized supplement of N in M. Thus M is a CWGS-module. □

Recall that a right R-module M is called *singular*, if $Z(M) = M$, where $Z(M) = \{m \in M : mI = 0,$ for some essential right ideal I of $R\}$ and *nonsingular* if $Z(M) = 0$. A ring R is called *right nonsingular* if R_R

is nonsingular and *singular* if R_R is singular. Let R be a ring, then R is right nonsingular, if and only if all right projective modules are nonsingular.

Let M be a nonsingular module and $N \leq_c M$, then $N \cap L \leq_c L$ for any submodule L of M. In fact, since M / N is nonsingular, so $(L + N)/N \cong L / (L \cap N)$ is nonsingular.

Corollary 3.8. *Let $M = M_1 + M_2$ be a nonsingular module such that M_1 is a CWGS-module and M_2 is any R-module. Then M is a CWGS-module, if and only if every closed submodule N of M with M_2 not contained in N has a weak generalized supplement.*

Theorem 3.9. *Let $M = M_1 \oplus M_2$ be a distributive module. Then M is a CWGS-module, if and only if each M_i, $i \in \{1, 2\}$, is a CWGS-module.*

Proof. Let L be any closed submodule of M. Then for each i, $i \in \{1, 2\}$, $L \cap M_i$ is closed in M_i. In fact, suppose that $L \cap M_1 \leq_e K \leq M_1$. Since $M_2 \cap L \leq_e M_2 \cap L$ and M is distributive, we have that $L = (M_1 \cap L) \oplus (M_2 \cap L) \leq_e K \oplus (M_2 \cap L)$. Hence $L = (M_1 \cap L) \oplus (M_2 \cap L) = K \oplus (M_2 \cap L)$, because L is closed in M. So $K = L \cap M_1$ and $L \cap M_1$ is closed in M_1. Therefore, there is a submodule K_i of M_i such that $M_i = K_i + (L \cap M_i)$ and $(L \cap M_i) \cap K_i = L \cap K_i \leq Rad(M_i)$, where $i \in \{1, 2\}$. Hence $M = M_1 \oplus M_2 = K_1 \oplus K_2 + ((L \cap M_1) \oplus (L \cap M_2))$ $= K_1 \oplus K_2 + L$ and $L \cap (K_1 \oplus K_2) = (L \cap K_1) \oplus (L \cap K_2) \leq Rad(M_1)$ $\oplus Rad(M_2) = Rad(M_1 \oplus M_2) = Rad(M)$. Thus M is a CWGS-module. The converse holds by Proposition 3.3. □

Corollary 3.10. *Let* $M = \bigoplus_{i=1}^{n} M_i$ *be a duo module. Then M is a CWGS-module, if and only if each* M_i, $i \in \{1, \ldots, n\}$, *is a CWGS-module.*

A pair (M, f) is called a *generalized cover* of a module *N*, if *f* is an epimorphism from *M* to *N* such that $\operatorname{Ker} f \subseteq Rad(M)$.

Proposition 3.11. *Let* (M, f) *be a generalized cover of a module N such that N is a CWGS-module. Suppose that every nonzero closed submodule L of M contains Ker f. Then M is a CWGS-module.*

Proof. Let $f : M \to N$ be an epimorphism such that $\operatorname{Ker} f \subseteq Rad(M)$ and *N* be a CWGS-module. Let $0 \neq L \leq_c M$ and $f(L) \leq_e K \leq N$. Then $L = L + \operatorname{Ker} f = f^{-1}f(L) \leq_e f^{-1}(K)$. Hence $L = f^{-1}(K)$ and $f(L) = K$ is a closed submodule of *N*. Since *N* is a CWGS-module, $f(L)$ has a weak generalized supplement in *N*. Thus, there exists a submodule *T* of *N* such that $N = f(L) + T$ and $f(L) \cap T \subseteq Rad(N)$. So $M = L + f^{-1}(T)$ and $L \cap f^{-1}(T) \subseteq Rad(M)$ by [1, 9.15]. Therefore, *L* has a weak generalized supplement in *M*, i.e., *M* is a CWGS-module. □

4. Rings for which all Nonsingular Modules are CWGS-Modules

In this section, we study rings in which all nonsingular modules are CWGS-modules.

Proposition 4.1. *Let M be an R-module with* $Rad(M) = 0$. *Then the following statements are equivalent*:

(1) *M is a CWGS-module.*

(2) *M is extending.*

Proof. It is clear. □

Corollary 4.2. *Let R be a semiprimitive ring. Then the following statements are equivalent*:

(1) *R is a CWGS-ring.*

(2) *R is an extending ring.*

Any homomorphic image of a WGS-module is a WGS-module (see [12, Proposition 3.2]). For homomorphic image of a CWGS-module, we have:

Theorem 4.3. *Let M be a CWGS-module. Then any nonsingular image of M is also a CWGS-module.*

Proof. Let $f : M \to N$ be an epimorphism of modules with M a CWGS-module and N a nonsingular module. Let L be a closed submodule of N. Since N is nonsingular, $H = f^{-1}(L)$ is a closed submodule of M. As M is a CWGS-module, there exists a submodule K of M such that $M = K + H$ and $K \cap H \leq Rad(M)$. Hence $N = f(K) + f(H) = f(K) + L$. Since $\operatorname{Ker} f \leq H$, $f(K \cap H) = f(K) \cap f(H) = f(K) \cap L \leq f(Rad(M)) \leq Rad(N)$. Thus N is a CWGS-module. □

Corollary 4.4. *Let M be a CWGS-module such that $M / Rad(M)$ is nonsingular. Then $M / Rad(M)$ is extending.*

Remark 4.5. In Theorem 4.3, the nonsingularity of N is not necessary. For example, $\mathbb{Z}$ is a CWGS-module, for any prime p, $\mathbb{Z}_p \cong \mathbb{Z} / p\mathbb{Z}$ is a simple $\mathbb{Z}$-module and a CWGS-module. But $\mathbb{Z}_p$ is singular.

Corollary 4.6. *Let R be a right nonsingular ring. Then the following statements are equivalent*:

(1) *Every nonsingular right R-module is a CWGS-module.*

(2) *Every projective right R-module is a CWGS-module.*

A ring R is called a *right CWGS-ring*, if R_R is a CWGS-module.

Corollary 4.7. *Let R be a right nonsingular ring. Then the following statements are equivalent*:

(1) *R is a right CWGS-ring.*

(2) *Every nonsingular cyclic R-module is a CWGS-module.*

(3) *Every principal right ideal of R is a CWGS-ring.*

A ring R is called a *right V-ring*, if every simple right R-module is injective. Equivalently, a ring R is a right V-ring, if and only if $Rad(M) = 0$ for all right R-modules M [13].

Theorem 4.8. *Let R be a right nonsingular V-ring. Then the following statements are equivalent*:

(1) *Every nonsingular right R-module M is a CWGS-module.*

(2) *Every projective right R-module M is a CWGS-module.*

(3) *Every nonsingular right R-module M is extending.*

(4) *Every nonsingular right R-module M is projective.*

Proof. (1) $\Leftrightarrow$ (2) By Corollary 4.6.

(1) $\Leftrightarrow$ (3) By Proposition 4.1.

(2) $\Rightarrow$ (4) Let M be a nonsingular module. There is a projective module P, such that $M \cong P / N$ for some submodule N of P. Since P is nonsingular, N is a closed submodule of P. By (2), P is a CWGS-module. Hence P is extending by Proposition 4.1. Thus N is a direct summand of P and therefore M is projective.

(4) $\Rightarrow$ (1) Let M be a nonsingular module and N be a closed submodule of M. Then M / N is nonsingular and so M / N is projective by (4). Thus N is a direct summand of M. Hence M is extending and so M is a CWGS-module. □

Theorem 4.9. *Let R be a right V-ring. Then the following statements are equivalent*:

(1) *R is right nonsingular and every nonsingular right R-module M is a CWGS-module.*

(2) *R is right nonsingular and every projective right R-module M is a CWGS-module.*

(3) *R is right nonsingular and every nonsingular right R-module M is extending.*

(4) *R is right nonsingular and every nonsingular right R-module M is projective.*

(5) *R is left nonsingular and every nonsingular left R-module M is a CWGS-module.*

(6) *R is left nonsingular and every projective left R-module M is a CWGS-module.*

(7) *R is left nonsingular and every nonsingular left R-module M is extending.*

(8) *R is left nonsingular and every nonsingular left R-module M is projective.*

Proof. It follows from Theorem 4.8 and [3, Theorem 5.23]. □

Lemma 4.10. *Let U and K be submodules of M such that K is a weak generalized supplement of a maximal submodule N of M. If $K + U$ has a weak generalized supplement X in M, then U has a weak generalized supplement in M.*

Proof. Since X is a weak generalized supplement of $K + U$ in M, $M = K + U + X$ and $X \cap (K + U) \leq Rad(M)$. If $K \cap (X + U) \subseteq K \cap N \leq Rad(M)$, then $U \cap (K + X) \leq [(X \cap (K + U)) + (K \cap (X + U))] \leq Rad(M)$. Hence $K + X$ is a weak generalized supplement of U in M.

Suppose that $K \cap (X + U)$ is not contained in $K \cap N$. Since $K/(K \cap N) \cong (K + N)/N = M/N$, $K \cap N$ is a maximal submodule of K. Therefore, $(K \cap N) + (K \cap (X + U)) = K$. Since $K \cap N \leq Rad(M)$, $M = X + U + K = X + U + (K \cap N) + (K \cap (X + U)) = X + U$. As $U \cap X \leq (K + U) \cap X \leq Rad(M)$, X is a weak generalized supplement of U in M. □

The following theorem is an immediate consequence of this lemma:

Theorem 4.11. *Suppose that for any submodule U of M, there exists a submodule K of M, which is a weak generalized supplement of some maximal submodule N of M, such that* $K + U$ *is closed in M. Then M is a CWGS-module, if and only if M is a WGS-module.*

References

[1] F. W. Anderson and K. R. Fuller, Rings and Categories of Modules, Springer-Verlag, New York, 1992.

[2] A. W. Chatters and S. M. Khuri, Endomorphism rings of modules over nonsingular CS rings, J. London Math. Soc. 21(2) (1980), 434-444.

[3] K. R. Goodearl, Ring Theory, Nonsingular Rings and Modules, Marcel Dekker, Inc., New York and Basel, 1976.

[4] A. Harmanci, D. Keskin and P. F. Smith, On ⊕-supplemented modules, Acta Math. Hungar 83 (1999), 161-169.

[5] F. Kasch, Modules and Rings, Acad. Press, London, 1982.

[6] D. Keskin, On lifting modules, Comm. Algebra 28 (2000), 3427-3440.

[7] C. Lomp, On semilocal modules and rings, Comm. Algebra 27(4) (1999), 1921-1935.

[8] S. H. Mohamed and B. J. Müller, Continuous and Discrete Modules, London Math. Soc. Lecture Notes Series 147, Cambridge, University Press, 1990.

[9] S. Nakahara, A generalization of semiperfect modules, Osaka J. Math. 20 (1983), 43-50.

[10] T. Takeuchi, Coranks of a quasi-projective module and its endomorphism ring, Glasgow Math. J. 36 (1994), 381-383.

[11] K. Varadarajan, Modules with supplements, Pacific J. Math. 82 (1979), 559-564.

[12] Y. Wang and N. Ding, Generalized supplemented modules, Taiwanese J. Math. 10 (2006), 1589-1601.

[13] R. Wisbauer, Foundations of Module and Ring Theory, Gordon and Breach, Reading, 1991.

[14] R. Wisbauer, Module and Algebras: Bi-module Structure and Group Actions on Algebras, Pitman Monographs and Surveys in Pure and Applied Mathematics 81 (1996).

[15] W. Xue, Characterizations of semiperfect and perfect rings, Pub. Math. 40 (1996), 115-125.

[16] Zeng Qing-yi and Shi Mei-hua, On closed weak supplemented modules, J. Zhejiang Univ. Sci. A 7(2) (2006), 210-215.

■